Roseville College | An Anglican School For Girls

Student Number ____________

2015

**YEAR 12**

**TRIAL EXAMINATION**

# Mathematics Extension 2

**General Instructions**

- Reading time - 5 minutes
- Working time - 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- Show all necessary working in Questions 11-16

**Total marks - 100**

**Section I**

**10 marks**

- Attempt Questions 1-10
- Allow about 15 minutes for this section

**Section II**

**90 marks**

- Attempt Questions 11-16
- Allow about 2 hour 45 minutes for this section

## Section 1

**10 marks**

**Attempt Questions 1 – 10**

**Allow about 15 minutes for this section**

Use the multiple-choice answer sheet for Questions 1 – 10.

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1 $\dfrac{2i}{1-2i} = ?$

(A) $\dfrac{4}{3} - \dfrac{2}{3}i$

(B) $-\dfrac{4}{5} + \dfrac{2}{5}i$

(C) $\dfrac{4}{5} + \dfrac{2}{5}i$

(D) $-\dfrac{4}{3} - \dfrac{2}{3}i$

2 The eccentricity of the hyperbola $\dfrac{x^2}{4k^2} - \dfrac{y^2}{k^2} = 1$, where $k$ is a positive constant, is?

(A) $\dfrac{\sqrt{3}}{2}$

(B) $2$

(C) $\dfrac{\sqrt{5}}{2}$

(D) $\sqrt{5}$

3 The value of $\int_{-1}^{2} \frac{1}{x^2+2x+10}\,dx$ is?

(A) $\frac{\pi}{12}$

(B) $\frac{\pi}{4}$

(C) $\frac{\pi}{36}$

(D) None of the above.

4 The gradient of the curve $xy - x^2 + 3 = 0$ at the point when $x = 1$ is:

(A) $-4$ (B) $-1$ (C) $1$ (D) $4$

5 The region bounded by the curves $y = x^2$ and $y = x^3$ in the first quadrant is rotated about the $y$–axis. The volume of the solid of revolution formed can be found using:

(A) $V = \pi\int_0^1 \left(y^{\frac{1}{3}} - y^{\frac{1}{2}}\right)\,dy$ (B) $V = \pi\int_0^1 \left(y^{\frac{1}{2}} - y^{\frac{1}{3}}\right)\,dy$

(C) $V = \pi\int_0^1 \left(y^{\frac{2}{3}} - y\right)\,dy$ (D) $V = \pi\int_0^1 (x^4 - x^6)\,dx$

6 What is the remainder when $P(x) = x^3 + x^2 - x + 1$ is divided by $(x - 1 - i)$?

(A) $-3i - 2$ (B) $3i - 2$ (C) $3i + 2$ (D) $2 - 3i$

7 The value of $\lim_{n\to\infty}\left[n\sin\left(\frac{2\pi}{n}\right)\right]$ is?

(A) $\frac{1}{2\pi}$

(B) $1$

(C) $0$

(D) $2\pi$

8 Solve the inequality: $\frac{x+1}{x-3} \leq \frac{x+3}{x-2}$

(A) $x < 2$ and $x > 3$

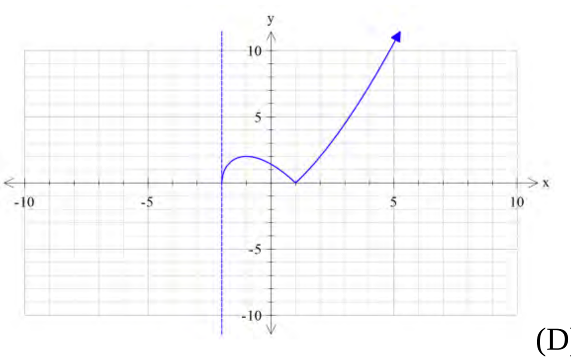
(B) $x < 2$ and $3 < x \leq 7$

(C) $2 < x < 3$

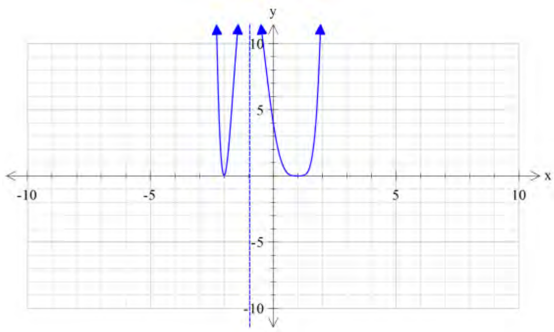
(D) $2 < x < 3$ and $x \geq 7$

9 Which of the diagrams below best represents $y = \sqrt{f(x)}$

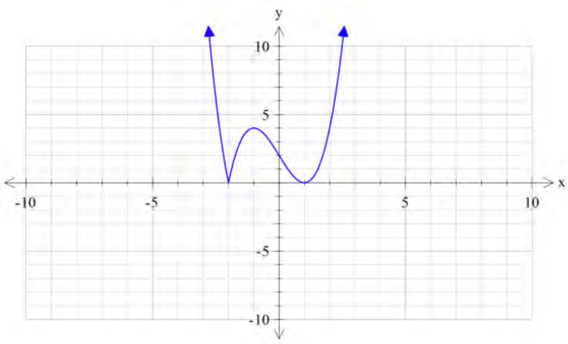
(A)



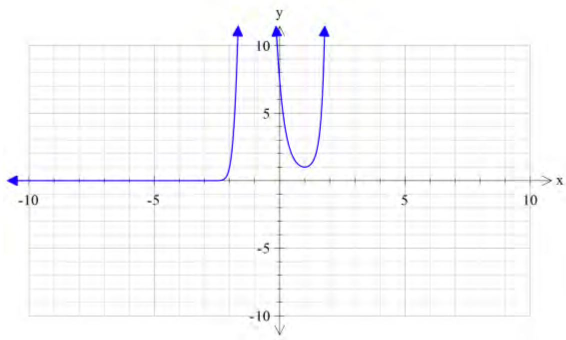
(B)



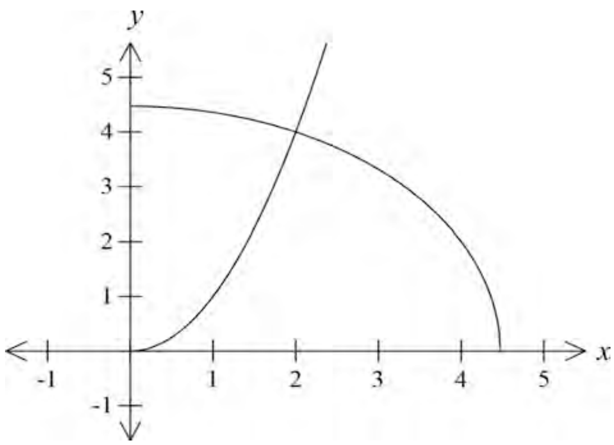
(C)



(D)



10. A solid is formed when the region bounded by the curves $y = x^2$, $y = \sqrt{20 - x^2}$ and the $y$-axis is rotated about the $y$-axis.




What is the correct expression for the volume of this solid using the method of cylindrical shells?

(A) $V = \int_0^2 2\pi\left(\sqrt{20 - x^2} - x^2\right)dx$

(B) $V = \int_0^2 2\pi x\left(\sqrt{20 - x^2} - x^2\right)dx$

(C) $V = \int_0^2 2\pi\left(x^2 - \sqrt{20 - x^2}\right)dx$

(D) $V = \int_0^2 2\pi x\left(x^2 - \sqrt{20 - x^2}\right)dx$

## End of Section I

## Section II

**90 marks**

**Attempt Questions 11 – 16**

**Allow about 2 hours and 45 minutes for this section**

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 – 16 your responses should include relevant mathematical reasoning and/or calculations.

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**Question 11** (15 marks) Use a SEPARATE writing booklet.

(a) (i) Write $(1+2i)^2$ in the form $x+iy$ where $x$ and $y$ are real. **1**

(ii) Solve $z^2 = -12+16i$. Write your answer in the form $x+iy$. **2**

(b) Evaluate $\displaystyle\int_0^1 \frac{2x+1}{x^2+1}\,dx$. **3**

(c) (i) Find $A$ and $B$ such that $\displaystyle\frac{4}{4-x^2} \equiv \frac{A}{2-x} + \frac{B}{2+x}$. **2**

(ii) Hence find $\displaystyle\int \frac{4}{4-x^2}\,dx$ **2**

(d) The equation $2x^3+4x-3=0$ has roots $x=\alpha, x=\beta$ and $x=\gamma$. **2**

Find a polynomial equation with roots $x=2\alpha, x=2\beta$ and $x=2\gamma$.

(e) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \cos\theta\sqrt{1+\sin\theta}\,d\theta$. **3**

**End of Question 11**

**Question 12** (15 marks) Use a SEPARATE writing booklet.

(a) The equation $x^4 + 2x^3 - 7x^2 - 20x - 12 = 0$ has a double root. Find this root and hence solve this equation. **3**

(b) Prove that the condition for the line $y = mx + c$ to touch the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is: **4**

$$c^2 = b^2 + a^2m^2$$

(c) Given that $1 + i$ is a root of the equation $z^2 + (a + 2i)z + (5 + ib) = 0$ where **a** and **b** are real, determine the values of **a** and **b**. **3**

(d) (i) Find the equation of the tangent at the point $P\left(ct, \frac{c}{t}\right)$ on the rectangular hyperbola $xy = c^2$. **2**

(ii) Find the coordinates of A and B where this tangent cuts the *x* and *y* axis respectively. **2**

(iii) Prove that the area of the triangle OAB is a constant. (Where O is the origin). **1**

## End of Question 12

**Question 13** (15 marks) Use a SEPARATE writing booklet.

(a) Let $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$, where $n$ is an integer and $n \geq 0$.

(i) Show that $I_n + I_{n-2} = \dfrac{1}{n-1}$. **3**

(ii) Hence find $\int_0^{\frac{\pi}{4}} \tan^4 x \, dx$. **2**

(b) $P\left(a\cos\theta, b\sin\theta\right)$, where $0 < \theta < \dfrac{\pi}{2}$, is a point on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

(i) Show that the tangent at $P$ has equation **2**

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1.$$

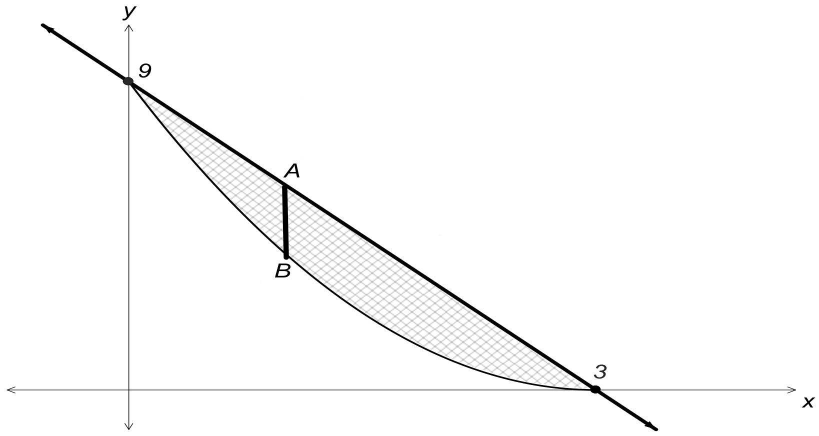
(ii) If $L$ is the distance of the tangent from the origin $O$ **3**

show that $L > \dfrac{ab}{\sqrt{a^2 + b^2}}$.

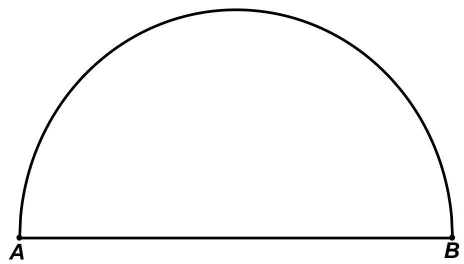
**Question 13 continues over the page**

**Question 13 continued**

(c)




Cross-section with base on *AB*




The diagram above shows the region enclosed by the parabola $y=(x-3)^2$ and the line $3x+y=9$. The region forms the base of a solid.

When the solid is sliced perpendicular to the *x*-axis, each cross-section is a semi-circle with diameter across the region.

A typical cross-section is shown above.

(i) If the solid is sliced along the line $x=a$, show that the area of the cross-section is $A=\frac{\pi}{8}a^2(3-a)^2$, where $0\le a\le 3$. **2**

(ii) Find the volume of the solid. **3**

## End of Question 13

**Question 14** (15 marks) Use a SEPARATE writing booklet.

(a) Using the substitution $t = \tan\frac{x}{2}$, or otherwise, evaluate **3**

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x}\,dx$$

(b) A sequence is defined such that $u_1 = 1, u_2 = 1$ and $u_n = u_{n-1} + u_{n-2}$ for $n \geq 3$. **4**

Prove by induction that $u_n < \left(\frac{7}{4}\right)^n$ for integers $n \geq 1$.

(c) A variable point $P(x, y)$ moves so that its distance from $(0,1)$ is one-half its distance from $y = 4$. Find, and describe the locus of $P$. **3**

(d) If $a > 0$, $b > 0$, $c > 0$ and $a + b + c = 1$ show that $(1-a)(1-b)(1-c) \geq 8abc$. **2**

(e) Show that $\int e^{-x}\cos x\,dx = \frac{1}{2}(\sin x - \cos x) + C$ **3**

## End of Question 14

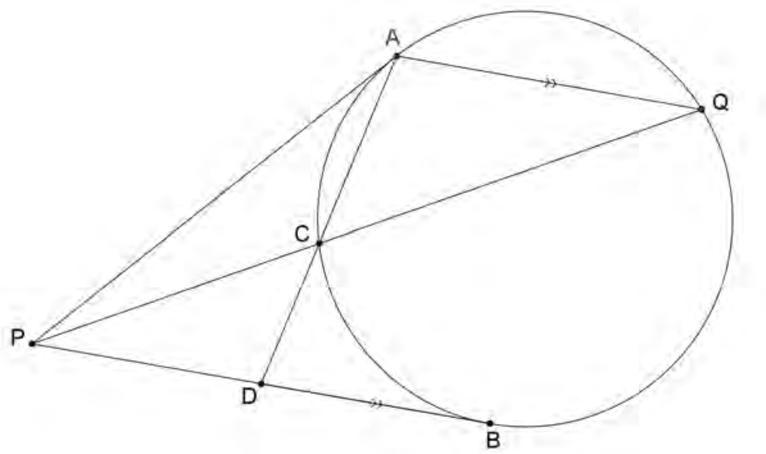
**Question 15** (15 marks) Use a SEPARATE writing booklet.

(a) In the diagram below, $PA$ and $PB$ are tangents to the circle. The chord $AQ$ is parallel to the tangent $PB$. $PCQ$ is a secant to the circle and chord $AC$ produced meets $PB$ at $D$.

(i) Show that $\Delta CDP$ is similar to $\Delta PDA$. **2**

(ii) Hence show that $PD^2 = AD \times CD$. **1**

(iii) Hence, or otherwise, prove that $AD$ bisects $PB$. **2**




(b) (i) Three identical balls are to be placed randomly in three trays. **2**

Each ball is equally likely to be placed in any one of the trays.

Show that the probability that exactly one of the trays is empty

is $\dfrac{2}{3}$.

(ii) The above process is repeated with $n$ identical balls (where $n \geq 2$) **2**

and $n$ trays. Write an expression in terms of $n$ for the probability that

exactly one tray is empty.

**Question 15 continues over the page**

**Question 15 continued**

(c) Use the method of cylindrical shells to find the volume of the solid generated by revolving the region enclosed by $y = 3x^2 - x^3$ and the $x$ axis around the $y$-axis. **3**

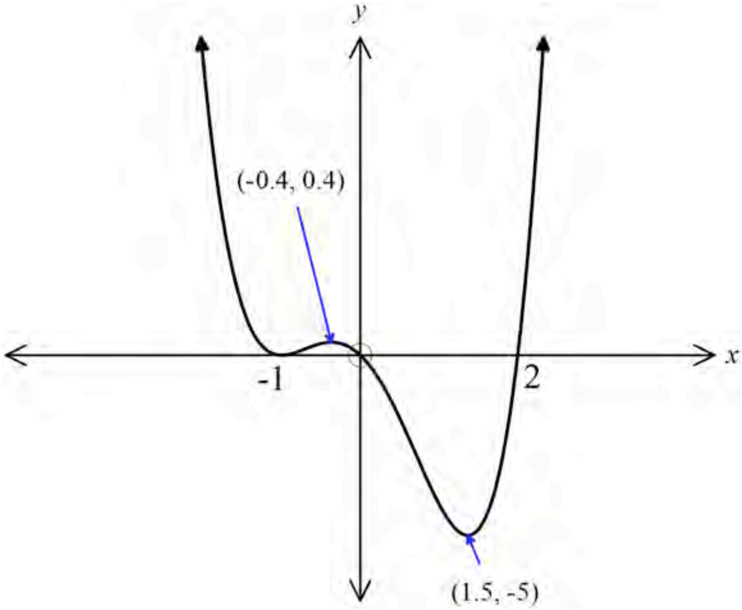
(d) By taking slices perpendicular to the axis of rotation, find the volume when

The area bounded by the parabola $y=6x-x^2$ and the $x$ axis is rotated about the line $y=10$ **3**

## End of Question 15

**Question 16** (15 marks) Use a SEPARATE writing booklet.

(a) The graph of $y = f(x)$ is shown below.




Draw separate sketches for each of the following:

(i) $y = |f(x)|$ **2**

(ii) $y = \frac{1}{f(x)}$ **2**

(iii) $y^2 = f(x)$ **2**

(iv) $y = e^{f(x)}$ **2**

(b) The function $F(p)$ is defined as $F(p) = \lim_{t \to \infty} \int_0^t x^{p-1} e^{-x} dx$, for $p > 0$.

(i) Show that $F(1) = 1$. **2**

(ii) Use integration by parts to show $F(p+1) = pF(p)$. **3**

(iii) Hence find $F(n)$ for integers $n \geq 1$. **2**

**End of paper.**

**STANDARD INTEGRALS**

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n\neq -1;\quad x\neq 0,\text{ if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a\neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a\neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a\neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a\neq 0$$

$$\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax,\quad a\neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a\neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a>0,\quad -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\quad x>0$

Suggested Solutions | Comments

Question MULTIPLE CHOICE

(1) $\frac{2i}{1-2i} \times \frac{1+2i}{1+2i} = \frac{2i-4}{1+4}$

$= -\frac{4}{5} + \frac{2i}{5}$

$\boxed{B}$

(2) $\frac{x^2}{4k^2} - \frac{y^2}{k^2} = 1$

$k^2 = 4k^2(e^2-1)$

$\frac{1}{4} = e^2 - 1$

$e^2 = \frac{5}{4}$

$e = \frac{\sqrt{5}}{2}$

$\boxed{C}$

(3) $\int_{-1}^{2} \frac{1}{x^2+2x+10}\,dx = \int_{-1}^{2} \frac{1\,dx.}{x^2+2x+1+9}$

$= \int_{-1}^{2} \frac{1}{(x+1)^2+9}\,dx = \frac{1}{3}\tan^{-1}\frac{x+1}{3}$

$= \frac{1}{3}\left[\tan^{-1}1 - 0\right]$

$= \frac{1}{3} \times \frac{\pi}{4} = \frac{\pi}{12}$

$\boxed{A}$

(4) $1.y + 1.\frac{dy}{dx}.x - 2x = 0$

$x = 1$
$y = -2$

at $(1,-2)$

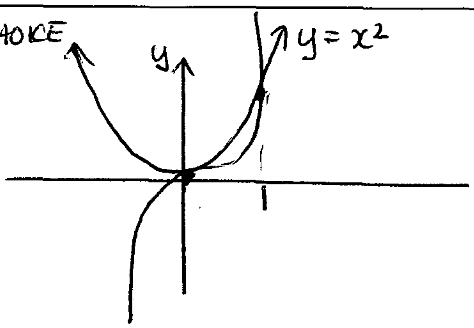
$-2 + \frac{dy}{dx} - 2 = 0$

$\frac{dy}{dx} = 4$

$\boxed{D}$

Question MULTCHOICE

(5)




$V = \pi \int x^2\, dy$

$V = \pi \int_0^1 (y^{1/3})^2 - (y^{1/2})^2$

$= \pi \int_0^1 y^{2/3} - y \, dy$

$\boxed{C}$

(6) $P(x) = x^3 + x^2 - x + 1$ $(x - 1 - i)$

$x - (1+i)$.

$P(1+i) = (1+i)^3 + (1+i)^2 - (1+i) + 1$

$= \cancel{1} + 3i - \cancel{3} - i + \cancel{1} + 2i - \cancel{1} - \cancel{1} - i + \cancel{1}$

$= -2 + 3i$

$\boxed{B}$

(7) $\lim_{n \to \infty} \left[ n \sin\left(\frac{2\pi}{n}\right) \right]$ * Small angle. $\sin\left(\frac{2\pi}{n}\right) = \frac{2\pi}{n}$.

$= \left[ \cancel{n} \times \frac{2\pi}{\cancel{n}} \right]$

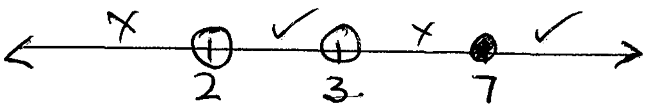
$\boxed{D}$

(8) Solve $(x+1)(x-2) = (x+3)(x-3)$

$\cancel{x^2} - x - 2 = \cancel{x^2} - 9$.

$-x = -7$

$x = 7$




for $x = 0$ $\cancel{(1)(-2)}$ $\frac{1}{-3} \leq \frac{3}{-2}$ $\frac{1}{3} \geq \frac{3}{2}$ ✗

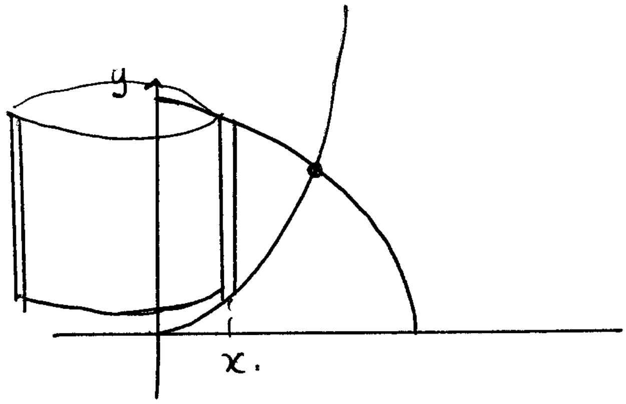
$2 < x < 3$, $x \geq 7$

$\boxed{D}$

Question MULTIPLE CHOICE

(9) $y = \sqrt{f(x)}$ $\boxed{A}$

(10)




$2\pi r = 2\pi x$

$A = 2\pi x(\sqrt{20-x^2} - x^2)$

$y_1 - y_2$

$\sqrt{20-x} - x^2$

$$V = \int_0^2 2\pi x(\sqrt{20-x^2} - x^2)\,dx$$

$\boxed{B}$

Question 11

(a) (i) $(1+2i)^2 = 1+4i-4$

$= \boxed{-3+4i}$ $\boxed{1}$

(ii) $z^2 = -12+16i$

$(x+iy)^2 = -12+16i$

$x^2+2xyi-y^2 = -12+16i$

$x^2-y^2 = -12$ --- ①

$2xy = 16$

$xy = 8 \rightarrow y = \frac{8}{x}$

$\therefore$ sub into ①

$x^2 - \frac{64}{x^2} = -12$

$x^4 - 64 = -12x$

$x^4 + 12x - 64 = 0$

$(x^2+16)(x^2-4) = 0$ $x$ is real.

$\therefore x = \pm 2$

$y = \pm 4$

$\therefore \boxed{z = 2-4i, -2-4i}$ $\boxed{2}$

Question 11

(b) $\int_0^1 \frac{2x+1}{x^2+1} = \int_0^1 \frac{2x}{x^2+1} + \frac{1}{x^2+1}\,dx$

$= \ln(x^2+1)\Big]_0^1 + \tan^{-1}x\Big]_0^1$

$= \ln(2) - \ln(1) + \tan^{-1}1 - 0$

$= \boxed{\ln 2 + \frac{\pi}{4}}$

3

(c) (i) $\frac{4}{4-x^2} = \frac{A}{2-x} + \frac{B}{2+x}$

$4 = A(2+x) + B(2-x)$.

for $x = -2$

$4 = B(4) \rightarrow \boxed{B = 1}$

for $x = 2$

$4 = A(4) \rightarrow \boxed{A = 1}$

2

(ii) $\int \frac{4}{4-x^2}\,dx = \int \frac{1}{2-x} + \frac{1}{2+x}\,dx$

$= -\ln(2-x) + \ln(2+x) + c$

$= \boxed{\ln\left|\frac{2+x}{2-x}\right| + c}$

2

Suggested Solutions | Comments

Question 11

(d) $2x^3+4x-3=0. \quad \alpha, \beta, \gamma$

$$x = 2\alpha$$
$$\alpha = \frac{x}{2}$$

$$\therefore \frac{2x^3}{8} + \frac{4x}{2} - 3 = 0$$

$$\boxed{x^3 + 8x - 12 = 0}$$

2

(e) $\displaystyle\int_0^{\frac{\pi}{2}} \cos\theta\sqrt{1+\sin\theta}\, d\theta$ let $u = 1+\sin\theta$, $du = \cos\theta\, d\theta$

$$= \int_1^2 u^{\frac{1}{2}}\, du$$

$$= \left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^2$$

$$= \frac{2}{3}\left[2\sqrt{2} - 1\right]$$

$$= \boxed{\frac{4\sqrt{2}-3}{3}}$$

3

**Question 12**

(a) $P(x) = x^4 + 2x^3 - 7x^2 - 20x - 12$

$P'(x) = 4x^3 + 6x^2 - 14x - 20.$

$P'(-2) = 0 \quad P(-2) = 0$

$\therefore$ Double root is $2$.

① Finding double root $x = 2$

$(x+2)^2 = x^2 + 4x + 4$

$$\begin{array}{r|l} & x^2 - 2x - 3 \\ \hline x^2+4x+4 & x^4 + 2x^3 - 7x^2 - 20x - 12 \\ & x^4 + 4x^3 + 4x^2 \\ & -2x^3 - 11x^2 - 20x - 12 \\ & -2x^3 - 8x^2 - 8x \\ & -3x^2 - 12x - 12 \\ & -3x^2 - 12x - 12 \end{array}$$

① Long division or equivalent

① Finding other roots.

$P(x) = (x+2)^2(x-3)(x+1)$

$$\boxed{x = -2, -2, 3, -1}$$

3

(b) $y = mx + c \qquad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$\therefore \dfrac{x^2}{a^2} + \dfrac{(mx+c)^2}{b^2} = 1$

$b^2x^2 + a^2m^2x^2 + 2a^2mcx + a^2c^2 = a^2b^2$

$(b^2 + a^2m^2)x^2 + 2a^2mcx + a^2c^2 - a^2b^2 = 0$

For $\Delta = 0$

① attempting to solve simultaneously

① $\Delta = 0$ or equivalent

① Algebraic manipulation with minor error.

**Question 12**

$$4a^4m^2c^2 - 4(b^2+a^2m^2)(a^2c^2-a^2b^2) = 0$$

$$\cancel{4a^4m^2c^2} - 4a^2b^2c^2 + 4a^2b^4 - \cancel{4a^4m^2c^2} + 4a^4m^2b^2 = 0$$

$$-4a^2b^2c^2 + 4a^2b^4 + 4a^4m^2b^2 = 0$$

① answer.

$$-c^2 + b^2 + a^2m^2 = 0$$

$$\therefore c^2 = b^2 + a^2m^2$$

as required

$\boxed{4}$

(c) $z^2 + (a+2i)z + (5+ib) = 0$

sub in $z = 1+i$

$$(1+i)^2 + (a+2i)(1+i) + (5+ib) = 0$$

① subbing into equation

$$\cancel{1}+2i-\cancel{1} + a + ai + 2i - 2 + 5 + ib = 0$$

① correct expansion

$$a+3 + 2i + ai + 2i + bi = 0$$

Equate real & imaginary parts

① Equating real / imaginary parts

$$a+3=0$$
$$\therefore a=-3$$

$$4+a+b=0$$
$$4-3+b=0$$
$$1+b=0$$
$$b=-1$$

$\boxed{3}$

$$\therefore \boxed{a=-3,\ b=-1}$$

Question 12

(d) (i) $xy = c^2$

$y = c^2 x^{-1}$

$y' = -c^2 x^{-2}$

$\therefore m_T = \frac{-c^2}{c^2 t^2}$

$= -\frac{1}{t^2}$

$\therefore y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$

$t^2 y - ct = -x + ct$.

$\boxed{x + t^2 y = 2ct}$

① correct gradient with correct working

① correct equation

2

(ii) at A $y = 0$

$x = 2ct$ $\therefore$ $\boxed{A = (2ct, 0)}$

at B $x = 0$

$t^2 y = 2ct$

$y = \frac{2c}{t}$ $\therefore$ $\boxed{B = (0, \frac{2c}{t})}$

① $A = (2ct, 0)$

① $B = (0, \frac{2c}{t})$

2

(iii) $A = \frac{1}{\cancel{2}} \times \cancel{2}c\cancel{t} \times \frac{2c}{\cancel{t}}$

$\boxed{A = 2c^2}$

which is a constant.

1

Question 13

(a) $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$

(i) $I_n + I_{n-2} = \frac{1}{n-1}$

$$\int_0^{\frac{\pi}{4}} \tan^n x \, dx = \int_0^{\frac{\pi}{4}} \tan^2 x \tan^{n-2} x \, dx$$

$$= \int_0^{\frac{\pi}{4}} (\sec^2 x - 1) \tan^{n-2} x \, dx$$

$$I_n = \int_0^{\frac{\pi}{4}} \sec^2 x \tan^{n-2} x \, dx - \int_0^{\frac{\pi}{4}} \tan^{n-2} x \, dx$$

$u = \tan x$

$du = \sec^2 x \, dx$.

$$I_n = \int_0^1 u^{n-2} du - I_{n-2}$$

$$I_n = \left[\frac{u^{n-1}}{n-1}\right]_0^1 - I_{n-2}$$

$$I_n = \frac{1}{n-1} - 0 - I_{n-2}$$

$$\therefore I_n + I_{n-2} = \frac{1}{n-1}$$

as required.

Comments:

① correct set leading to correct answer

① correct integration (using parts)

① correct manipulation leading to answer

$\boxed{3}$

Question 13

(ii) $\int_0^{\frac{\pi}{4}} \tan^4 x \, dx$ $\int_0^{\frac{\pi}{4}} \tan^0 x \, dx$.

$= \left[\frac{\pi}{4} - 0\right]$

① correct sub into formula

$I_4 = \frac{1}{3} - I_2$

$I_4 = \frac{1}{3} - \left[\frac{1}{1} - I_0\right]$

$= \frac{1}{3} - \left[1 - \frac{\pi}{4}\right]$

$$\boxed{I_n = \frac{\pi}{4} - \frac{2}{3}}$$

① correct use of formula or finding $I_2$ and or $I_0$ or correct answer

$\boxed{2}$

(b) $P(a\cos\theta, b\sin\theta)$ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(i) $\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$

$\frac{2y}{b^2}\frac{dy}{dx} = -\frac{2x}{a^2}$

$\frac{dy}{dx} = -\frac{\cancel{2}x}{a^2} \times \frac{b^2}{\cancel{2}y}$

$m_T = -\frac{\cancel{a}\cos\theta \, b^{\cancel{2}}}{a^{\cancel{2}} \cancel{b}\sin\theta}$

$= -\frac{b\cos\theta}{a\sin\theta}$

① correct gradient with working

Question 13

$$\therefore y - b\sin\theta = -\frac{b\cos\theta}{a\sin\theta}(x - a\cos\theta)$$

$$a\sin\theta y - ab\sin^2\theta = -b\cos\theta x + ab\cos^2\theta$$

$$a\sin\theta y + b\cos\theta x = ab$$

$$\frac{\sin\theta y}{b} + \frac{\cos\theta x}{a} = 1$$

as required.

(1) correct manipulation to get required equation

$\boxed{2}$

(ii) $$d = \frac{|0+0-1|}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}}$$

$$= \frac{1}{\frac{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}{ab}}$$

(1) correct expression for $\perp$ distance

$$= \frac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

since $0 \leq \sin^2\theta \leq 1$, $0 \leq \cos^2\theta \leq 1$

(1) Realising to $0 \leq \sin^2\theta \leq 1$, $0 \leq \cos^2\theta \leq 1$

$$b^2\cos^2\theta + a^2\sin^2\theta < a^2 + b^2$$

$$\therefore \frac{1}{b^2\cos^2\theta + a^2\sin^2\theta} > \frac{1}{a^2+b^2}$$

$$\therefore \frac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}} > \frac{ab}{\sqrt{a^2+b^2}}$$

as required

(1) Relating the above to come up with required inequality

$\boxed{3}$

Question 13

(c) (i) $A = \frac{\pi}{2} r^2$

$$= \frac{\pi}{2}\left[\frac{(9-3x)-(x-3)^2}{2}\right]^2$$

$$= \frac{\pi}{2}\left[\frac{\cancel{9}-3x-x^2+6x-\cancel{9}}{2}\right]^2$$

$$= \frac{\pi}{2}\left[\frac{3x-x^2}{4}\right]^2$$

$$= \frac{\pi}{8}\left[x(3-x)\right]^2$$

at $x = a$

$$A = \frac{\pi}{8} a^2(3-a)^2$$

as required.

(ii) $V = \int_0^3 \frac{\pi}{8}(9a^2 - 6a^3 + a^4)\,da$

$$= \frac{\pi}{8}\left[3a^3 - \frac{3a^4}{2} + \frac{a^5}{5}\right]_0^3$$

$$= \frac{\pi}{8}\left[\frac{81}{10}\right]$$

$$\boxed{V = \frac{81\pi}{80}\ u^3}$$

Comments:

(1) correct r

(1) correct expression for A or correct expression with incorrect r.

$\boxed{2}$

(1) correct integral for V

(1) correct integration

(1) correct sub. and answer

$\boxed{3}$

Question 14

(a) $$\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x}\,dx$$

$t = \tan\frac{x}{2}$

$\frac{x}{2} = \tan^{-1} t$

$x = 2\tan^{-1} t$

$dx = \frac{2\,dt}{1+t^2}$

$$= \int_0^1 \frac{1}{1+\frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2}\,dt$$

- ① correct sub for $dx$ or $\cos x$

$$= \int_0^1 \frac{\cancel{1+t^2}}{1+\cancel{t^2}+1-\cancel{t^2}} \times \frac{2}{\cancel{1+t^2}}\,dt$$

- ① correct sub including limits

$$= \int_0^1 1\,dt$$

- ① Correct integration and answer

$$= \left. t \right]_0^1$$

$$= \boxed{1}$$

$\boxed{3}$

(b) $U_1 = 1$, $U_2 = 1$, $U_n = U_{n-1} + U_{n-2}$ $n \geq 3$

For $n = 1$

$U_1 = 1 < \left(\frac{7}{4}\right)^1$

$\therefore$ true for $n = 1$

For $n = 2$

$U_2 = 1 < \left(\frac{7}{4}\right)^2 \Rightarrow 1 < \frac{49}{16}$

$\therefore$ true for $n = 2$

① Prove true for $U_1$ and $U_2$

Question 14

Assume true for integers up to $n=k$

$\therefore \quad U_k < \left(\frac{7}{4}\right)^k$

① Assumption step - correctly stated or RTP step.

Prove true for $n=k+1$

R.T.P. $U_{k+1} < \left(\frac{7}{4}\right)^{k+1}$

① Set up of proof - step which could lead to correct solution

Now $U_{k+1} = U_k + U_{k-1}$

$$< \left(\frac{7}{4}\right)^k + \left(\frac{7}{4}\right)^{k-1}$$

$$= \left(\frac{7}{4}\right)^k + \left(\frac{7}{4}\right)^k \times \frac{4}{7}$$

$$= \left(\frac{7}{4}\right)^k \left(1 + \frac{4}{7}\right)$$

① correct manipulation and answer

$$= \left(\frac{7}{4}\right)^k \left(\frac{11}{7}\right)$$

$$< \left(\frac{7}{4}\right)^k \left(\frac{7}{4}\right)$$

$$= \left(\frac{7}{4}\right)^{k+1}$$

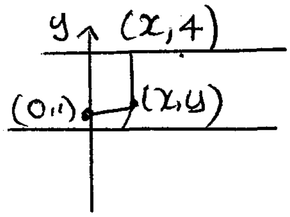
$\therefore \quad U_{k+1} < \left(\frac{7}{4}\right)^{k+1}$

$\therefore$ True by induction

$\boxed{4}$

**Question 14**

(c)




$$(x-0)^2+(y-1)^2 = \frac{1}{4}\left[(x-x)^2+(y-4)^2\right]$$

$$x^2+y^2-2y+1 = \frac{1}{4}\left[y^2-8y+16\right]$$

$$4x^2+4y^2-\cancel{8y}+4 = y^2-\cancel{8y}+16$$

$$4x^2+3y^2 = 12$$

$$\therefore \boxed{\frac{x^2}{3}+\frac{y^2}{4} = 12}$$

ellipse.

Comments: (1) correct set up or equivalent; (1) correct algebraic steps; (1) correct answer including description. $\boxed{3}$

(d) $a+b+c=1$ $\quad (1-a)(1-b)(1-c) \geqslant 8abc$.

$a = 1-b-c$
$b = 1-a-c$
$c = 1-a-b$

$$\text{LHS} = (1-1+b+c)(1-1+a+c)(1-1+a+b)$$
$$= (b+c)(a+c)(a+b)$$
$$\geqslant 2\sqrt{bc}\times 2\sqrt{ac}\times 2\sqrt{ab}$$
$$= 8\sqrt{a^2b^2c^2}$$
$$= 8abc$$

$b+c \geqslant 2\sqrt{bc}$
$a+c \geqslant 2\sqrt{ac}$
$a+b \geqslant 2\sqrt{ab}$.

as required.

Comments: (1) Attempt which could lead to a solution; (1) algebraic manipulation and answer. $\boxed{2}$

Suggested Solutions | Comments

Question 14

(e) $I_n = \int e^{-x} \cos x \, dx$.

$$I_n = -e^{-x}\cos x - \int -e^{-x}(-\sin x)\,dx$$

$$= -e^{-x}\cos x - \int e^{-x}\sin x\,dx$$

$$I_n = -e^{-x}\cos x - \left[-e^{-x}\sin x - \int -e^{-x}\cos x\,dx\right]$$

$$I_n = -e^{-x}\cos x + e^{-x}\sin x - I_n + c$$

$$2I_n = -e^{-x}\cos x + e^{-x}\sin x + c$$

$$= e^{-x}(\sin x - \cos x) + c$$

$$\therefore I_n = \frac{1}{2}e^{-x}(\sin x - \cos x) + c$$

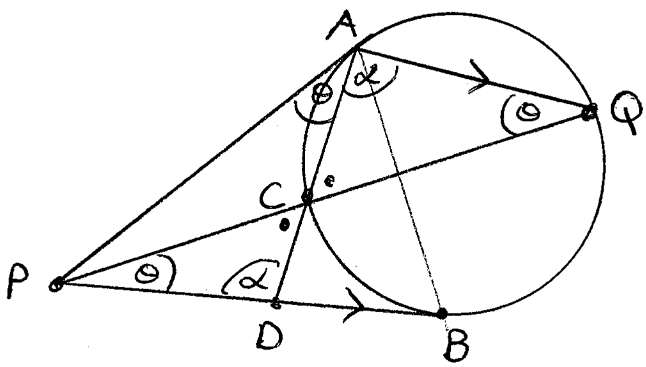
as required.

- ① correct use of integration by parts
- ① Second attempt of integration by parts or equivalent
- ① Algebraic manipulation and answer.

$\boxed{3}$

Question 15

(a)




(i) In Δs CDP and PDA

∠CDP = ∠PDA (common ∠)

∠PAD = ∠AQP (alt segment theorem)

∠AQP = ∠CPD (alt ∠'s AQ // PB)

∴ ∠PAD = ∠CPD (both = ∠AQP)

∴ ΔCDP ||| Δ PDA (AAA)

① Finding 1 equal ∠ or equivalent with reasons
① Finding 2nd angle with reasons. or equivalent

$\boxed{2}$

(ii) $\dfrac{PD}{AD} = \dfrac{CD}{PD}$

$\therefore (PD)^2 = AD \times CD$

as required

$\boxed{1}$

(iii) $(BD)^2 = AD \times CD$ (property of tangent and secants)

$\therefore (BD)^2 = (PD)^2 \rightarrow$ both $= AD \times CD$

$\therefore BD = PD$

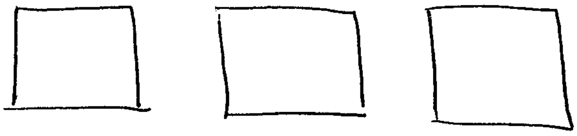
$\therefore$ AD bisects PB

① attempt to use correct theorem
① Equating $BD^2 = PD^2$

$\boxed{2}$

Question 15

(b)



(i) $P(\text{exactly 1 empty}) = \dfrac{{}^3C_2 \times 3!}{3^3}$

$= \dfrac{3 \times 6}{27}$

$= \dfrac{2}{3}$

as required

① Numerator or denominator correct or minor error in both.

① correct answer from previous step (within reason)

$\boxed{2}$

(ii) $P(\text{exactly 1 empty}) = \boxed{\dfrac{{}^nC_2 \times n!}{n^n}}$ *

$= \dfrac{\dfrac{n!}{(n-2)!\,2!} \times n!}{n^n}$

$= \dfrac{n! \times n!}{2(n-2)! \times n^n}$

$= \dfrac{n!\,n(n-1)}{2n^n}$

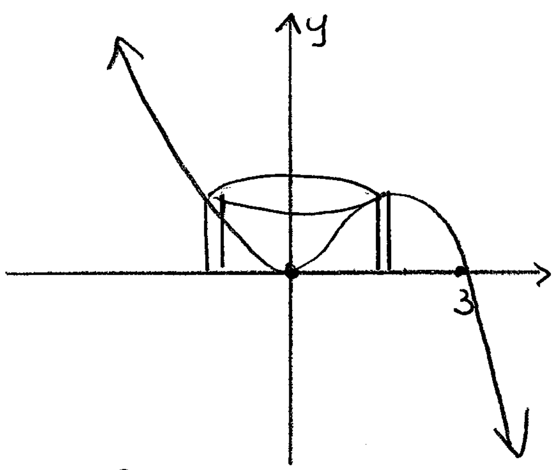
① Correct set up or establishing pattern from part (i)

① Any form of correct answer

$\boxed{2}$

**Question 15**

(c)




$y = 3x^2 - x^3$

$= x^2(3 - x)$

$2\pi r = 2 \times \pi \times x$

$\boxed{A = 2\pi x\,(3x^2 - x^3)}$ $y = 3x^2 - x^3$

① Correct height or radius or equivalent

$$\Delta V = 2\pi x\,(3x^2 - x^3)\,\Delta x$$

$$V = 2\pi \lim_{\Delta x \to 0} \sum_{0}^{3} 3x^3 - x^4\,\Delta x$$

$$V = 2\pi \int_0^3 3x^3 - x^4\,dx$$

① Correct set expression for A or correct set up for V from previous steps

$$= 2\pi \left[ \frac{3x^4}{4} - \frac{x^5}{5} \right]_0^3$$

$$= 2\pi \left[ \frac{243}{20} \right]$$

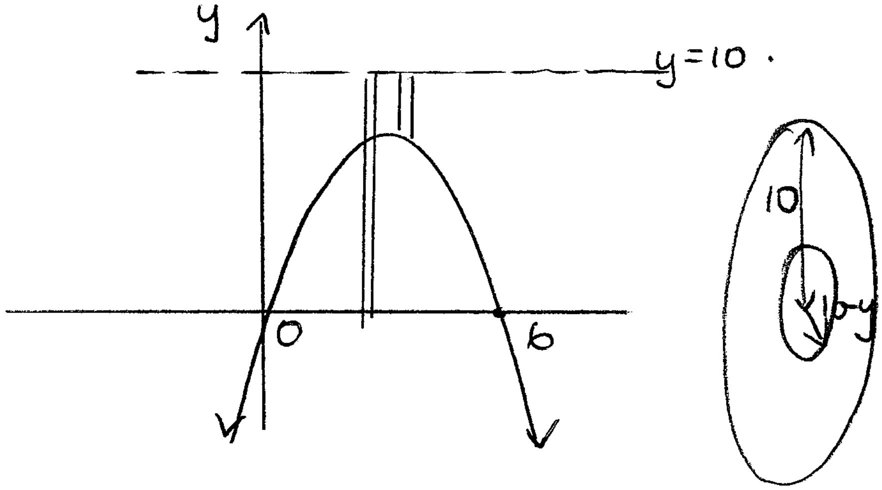
$$\boxed{V = \frac{243\pi}{10}\ u^3}$$

① Correct integration and answer

$\boxed{3}$

Question 15

(d)




$A = \pi[R^2 - r^2]$

$= \pi[10^2 - (10-y)^2]$

$A \quad \pi[100 - 100 + 20y - y^2]$

$= \pi[20y - y^2]$

$A(x) = \pi[20(6x - x^2) - (6x - x^2)^2]$

$= \pi[120x - 20x^2 - 36x^2 + 12x^3 - x^4]$

$= \pi[120x - 56x^2 + 12x^3 - x^4]$

$\Delta V = \pi[120x - 56x^2 + 12x^3 - x^4]\Delta x$

$V = \pi \lim_{\Delta x \to 0} \sum_{0}^{6} 120x - 56x^2 + 12x^3 - x^4 \Delta x$

$V = \pi \int_0^6 120x - 56x^2 + 12x^3 - x^4 \, dx$

$= \pi\left[60x^2 - \frac{56x^3}{3} + 3x^4 - \frac{x^5}{5}\right]_0^6$

$= \pi\left[\frac{2304}{5}\right]$

$\boxed{V = \frac{2304}{5}\pi u^2}$

① correct use of washer method

① Correct expression for A or ΔV or V.

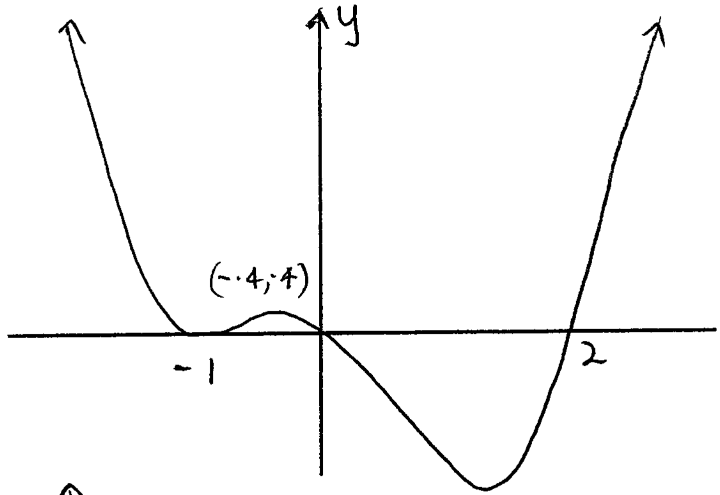
① Correct integration of appropriate function and answer

$\boxed{3}$

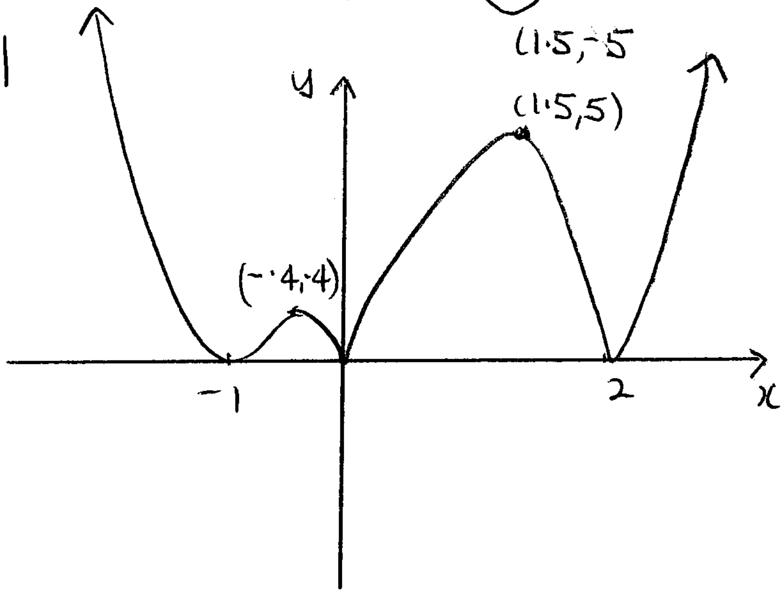
Suggested Solutions | Comments

Question 16

(a)

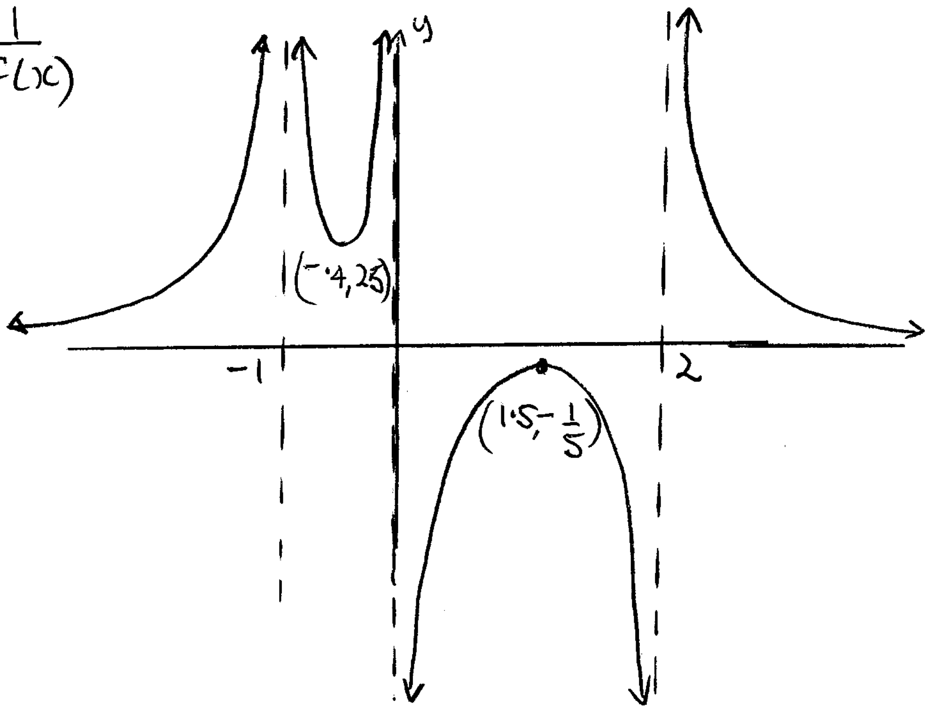



(i) $y = |f(x)|$




2

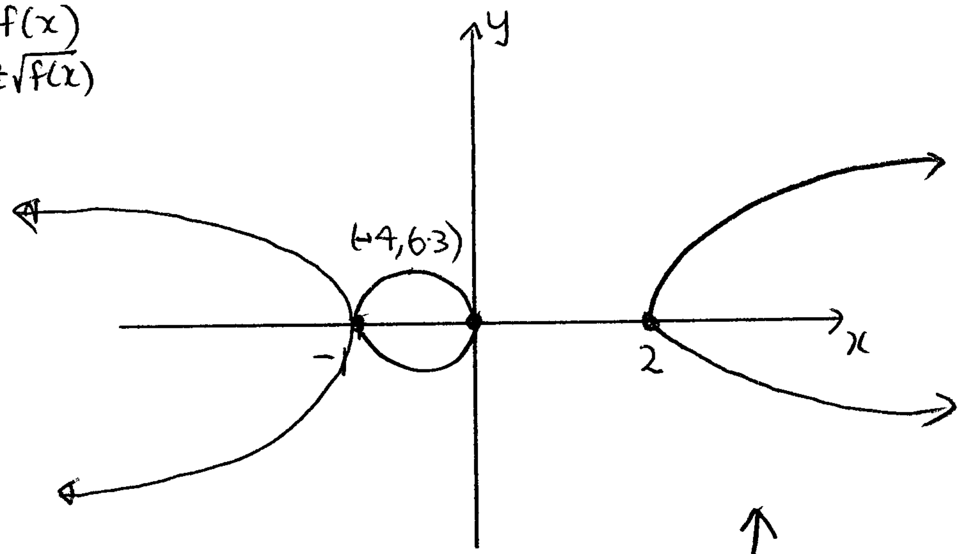
(ii) $y = \frac{1}{f(x)}$




2

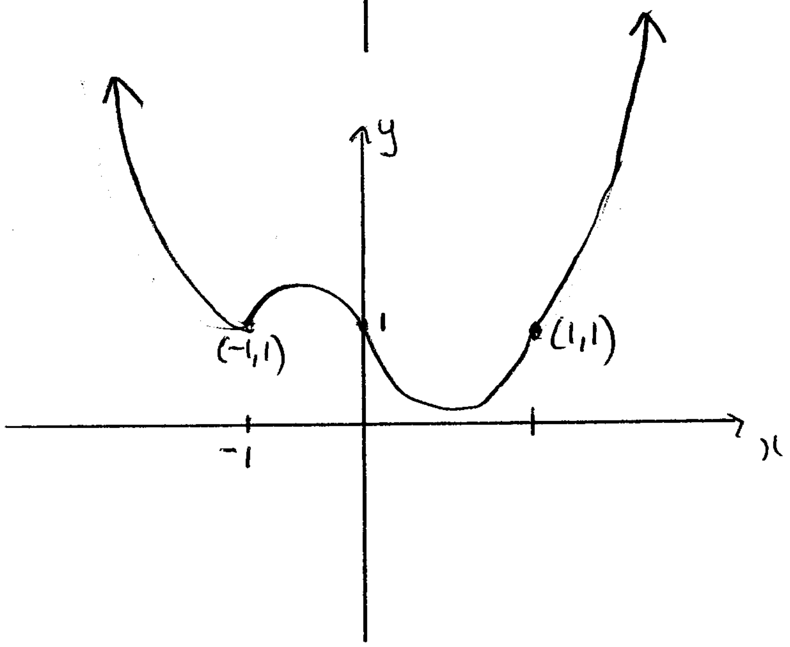
Question (16)

(iii) $y^2 = f(x)$
$y = \pm\sqrt{f(x)}$




$\boxed{2}$

(iv) $y = e^{f(x)}$




$\boxed{2}$

(b) $F(p) = \lim_{t\to\infty} \int_0^t x^{p-1} e^{-x}\,dx$

(i) $F(1) = \lim_{t\to\infty} \int_0^t x^0 e^{-x}\,dx$

$= \lim_{t\to\infty} \left[-e^{-x}\right]_0^t$

$= \lim_{t\to\infty} \left[-e^{-t} + 1\right]$

$= 1$

as required.

– ① correct integration and sub
– ① $\lim_{t\to\infty} e^{-t} = 0$.

$\boxed{2}$

Question (16)

(ii) $F(p+1) = pF(p)$.

$$\text{LHS} = \lim_{t\to\infty} \int_0^t x^{p+1-1} e^{-x}\, dx$$

$$= \lim_{t\to\infty} \int_0^t x^p e^{-x}\, dx$$

— ① correct expression for $F(p+1)$ or equivalent

$$= \lim_{t\to\infty} \left[ \left[-e^{-x} x^p\right]_0^t - \int_0^t p x^{p-1} - e^{-x}\, dx \right]$$

— ① correct use of int. parts

$$= \lim_{t\to\infty} \left[ -e^{-t} t^p - 0 + p\int_0^t x^{p-1} e^{-x}\, dx \right]$$

$$= \lim_{t\to\infty} \left[ 0 + p\int_0^t x^{p-1} e^{-x}\, dx \right]$$

— ① correct execution and sob in of $F(p)$

$$= p \times \lim_{t\to\infty} \int_0^t x^{p-1} e^{-x}\, dx$$

$\boxed{3}$

$$= pF(p)$$

$$= \text{RHS}.$$

iii

| Suggested Solutions | Comments |
|---|---|
| **Question 16** | |
| $F(n+1) = nF(n)$ | — 1 Showing any pattern from $F(n+1)=nF(n)$ |
| $\therefore F(n) = (n-1)F(n-1)$ | |
| $= (n-1)(n-2)F(n-2)$ | |
| $= (n-1)(n-2)(n-3)F(n-3)$ | — 1 Answer |
| $= (n-1)(n-2)(n-3)\ldots\ldots 1$ | |
| $\boxed{F(n) = (n-1)!}$ | $\boxed{2}$ |